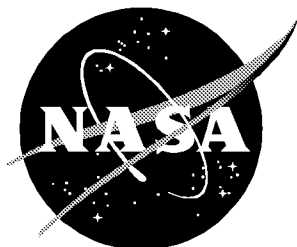# Generic Equations for Constructing Smooth Paths Along Circles and Tangent Lines With Application to Airport Ground Paths

L. Keith Barker
Langley Research Center, Hampton, Virginia

December 1998

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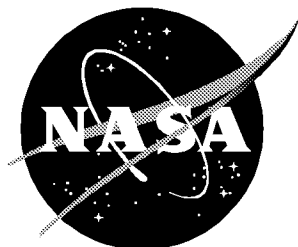
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Generic Equations for Constructing Smooth Paths Along Circles and Tangent Lines With Application to Airport Ground Paths

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Abstract

The primary purpose of this publication is to develop a mathematical model to describe smooth paths along any combination of circles and tangent lines. Two consecutive circles in a path are either tangent (externally or internally) or they appear on the same (lateral) or opposite (transverse) sides of a connecting tangent line. A path may start or end on either a segment or circle. The approach is to use mathematics common to robotics to design the path as a multilink manipulator. This approach allows a hierarchical view of the problem and keeps the notation manageable. A user simply specifies a few parameters to configure a path. Necessary and sufficient conditions automatically ensure the consistency of the inputs for a smooth path. Two example runway exit paths are given, and an angle to go assists in knowing when to switch from one path element to the next.

1. Introduction

A NASA goal is to improve the aviation system throughput in all weather conditions. One aspect of this problem is guidance along the runway and exit path. In this respect, coordinates along the centerline of a runway and the exit path must be known. Although others (refs. 1, 2, and 3) have modeled specific exit paths (taxiways) with linear segments and/or circles, a general formulation does not appear to exist for such paths.

This publication completely characterizes smooth paths along an arbitrary combination of circles and tangent lines. The approach herein is to configure the path as a multilink manipulator and use the mathematics common to robotics. This approach provides a ready-made high-level perspective of a complex system. A user simply specifies a few parameters in a computer program to configure a new path. Two examples of designing runway exit paths are given. Also, as a convenience, this publication introduces a parameter to tell when to switch from one path element to the next.

2. Symbols

A_j^i homogeneous transformation matrix from axis system i to axis system j or transform from circle C_i to circle C_j

a_i component of vector from C_{i-1} to C_i along X_i

C circle

ccw   counterclockwise

cir   circle

cw   clockwise

e_i circle classification, either exterior ($i = 1$) or inward ($i = -1$)

$(h_i, k_i, 0)$ center of circle C_i in general reference frame (X_G, Y_G, Z_G)

i, j index

L_G^{i+1} transformation from circle C_{i+1} to general reference frame with $l_{i+1} = 0$

l_i length of segment P_i, positive or negative

N number of circles in path

P_i tangent segment before circle C_i

p_i^{i-1} vector (in axis system $i - 1$) from axis system $i - 1$ to axis system i or vector from center of circle C_{i-1} to center of circle C_i

R_i radius of circle C_i

R_j^i Euler rotation matrix about Z_i from axis system i to axis system j

seg   segment

(X_G, Y_G, Z_G)	general reference frame
(X_i, Y_i, Z_i)	axis system rigidly attached to center of circle C_i
(X'_i, Y'_i, Z'_i)	axis system parallel to (X_G, Y_G, Z_G) at center of circle C_i
x_G, y_G, z_G	coordinates in general reference frame (X_G, Y_G, Z_G)
$x_{G,\text{exit}}, y_{G,\text{exit}}$	exit coordinates in general reference frame (X_G, Y_G, Z_G)
x_i, y_i, z_i	coordinates in circle axis system (X_i, Y_i, Z_i)
$(x'_i, y'_i, 0)$	coordinates in circle axis system (X'_i, Y'_i, Z'_i) parallel to (X_G, Y_G, Z_G)
$(x^*_i, y^*_i, 0)$	starting point of motion on circle C_i
$(x^\#_i, y^\#_i, 0)$	ending point of motion on circle C_i
Γ	azimuth of initial motion on first circle C_0 relative to X_G or angle between X_G and Y_0
θ_i	rotation angle about center of circle C_i or rotation about Z_{i-1} to rotate circle C_i about circle C_{i-1}
Λ_i	central angle of circle C_i, measured between current motion position and starting point for motion on circle; negative value means prior to starting point
λ_i	travel angle (central angle) on circle C_{i-1} about z_{i-1}
ϕ_i	central angle on C_i between X_i and radial ray to known point or radial projection angle
ϕ'_i	central angle on C_i between Y'_i and radial ray to known point or radial projection angle
$\psi_{G,i}$	azimuth on circle C_i relative to axis X_G
ψ_i	azimuth on circle C_i from north
ψ_{P_i}	azimuth on segment P_i from north
ψ_{X_G}	azimuth of axis X_G
ω_i	index of rotation, clockwise ($i = -1$) or counterclockwise ($i = 1$) on circle C_{i-1}

Subscripts:

max	maximum
target	target value on path

3. Circle Classification

A known path can be approximated by measuring a series of points along the path and then linearly connecting the points (line segments). However, this requires many points to approximate closely a curvilinear path, and even then, the slope of the path where the segments connect is discontinuous.

The method discussed in this publication constructs a smooth path by using a combination of circles and lines. Any two successive circles in a path are either tangent or connected by a common tangent line. A path may begin or end on a circle or line. A path is the composite of pieces of travel along circles and tangent lines. A user specifies information about the circles and segments to generate the path. If two immediately consecutive circles in a path are tangent, except for the tangent point, either (1) the second circle is external to the first (disjoint interiors), (2) the second circle is inside the first circle, or (3) the first circle is inside the second circle.

Any two successive circles in a path (say, first and second circles) are either tangent or connected by a single common tangent line. The second circle is either an exterior circle or inward circle with respect to the first circle.
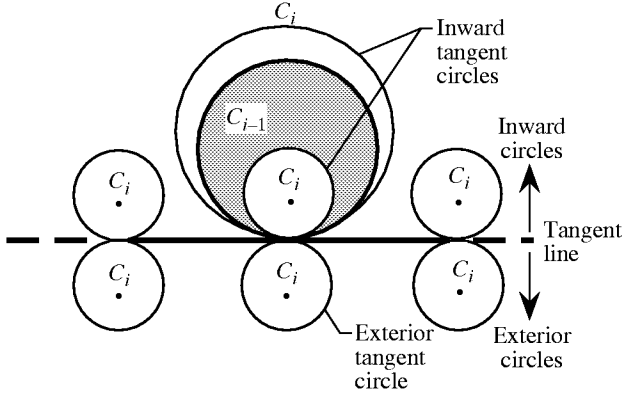
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Figure 1. Possible locations and classifications of second circle. C_{i-1} is first of two circles; C_i is second circle choice.

3.1. Exterior Circle

If two circles lie on opposite sides of a given tangent line, the second circle is an external (or transverse) circle with respect to the first, as shown in figure 1.

3.2. Inward Circle

If the first and second circles lie on the same side of a given common tangent line, the second circle is an inward (or lateral) circle with respect to the first circle, as shown in figure 1. Notice that the second circle can be inside the first or vice versa.

3.3. Parameter for Circle Type

Symbolically, the circle classification is

$$e_i = \left\{ \begin{array}{ll} 1 & (C_i \text{ exterior}) \\ -1 & (C_i \text{ inward}) \end{array} \right\} \qquad (1)$$

4. Modeling Path as Multilink Manipulator

A robot manipulator (ref. 4) is a series of joints connected by links; for example, the sketch in figure 2 shows a manipulator arm with three rotational joints (shoulder, elbow, and wrist). Each joint only moves the part of the manipulator beyond that joint; that is, the elbow joint rotates the forearm but the wrist joint does not. To configure a manipulator, one changes the joint angles.
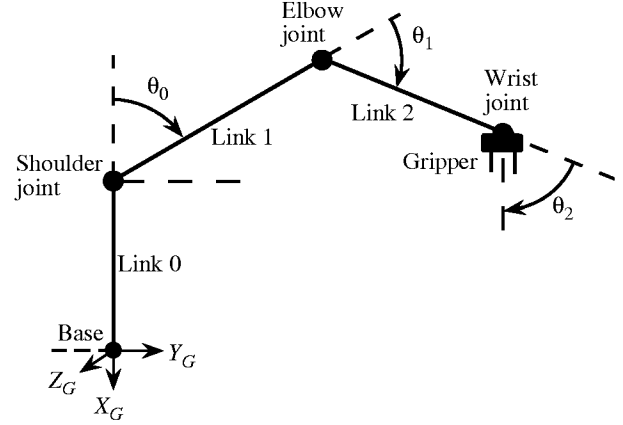

Figure 2. Multilink manipulator.

Designing a path is somewhat analogous to configuring a planar multilink manipulator if one thinks of the manipulator body as the path. Imagine, for example, that the manipulator in figure 2 is lying flat on a horizontal plane and think of the body of the manipulator as a road or path on this plane. A path of linear segments then follows. However, for a smooth path the segments should make a smooth transition into one another. Hence, imagine the joints as centers of circles and the links as tangent to these circles. Now, imagine moving along a path from the base of the manipulator toward the shoulder along link 0, which ends as a tangent to the shoulder circle. Proceed around the circumference of the shoulder circle for so many degrees (travel angle) and then travel toward the elbow along link 1, which is tangent to both the shoulder circle and the elbow circle, and so forth until the end of the path. Changing the joint angles changes the path. Really there are no joints on the path, but this publication relates the travel angle along the circular arcs to the joint angles to use mathematics common to robotics, which is the basic idea in this publication.

A known path can be approximated with a path along circles and tangent lines. To describe a path using circles and lines, one must know the radii of the circles, where the circles are located, the angular travel along each circle, and the length of any linear segment between circles.

4.1. Axis Systems

Figure 3 shows the initial layout of circles and linear segments to form a path. All axis systems are right-handed. As shown, all circles are exterior and all
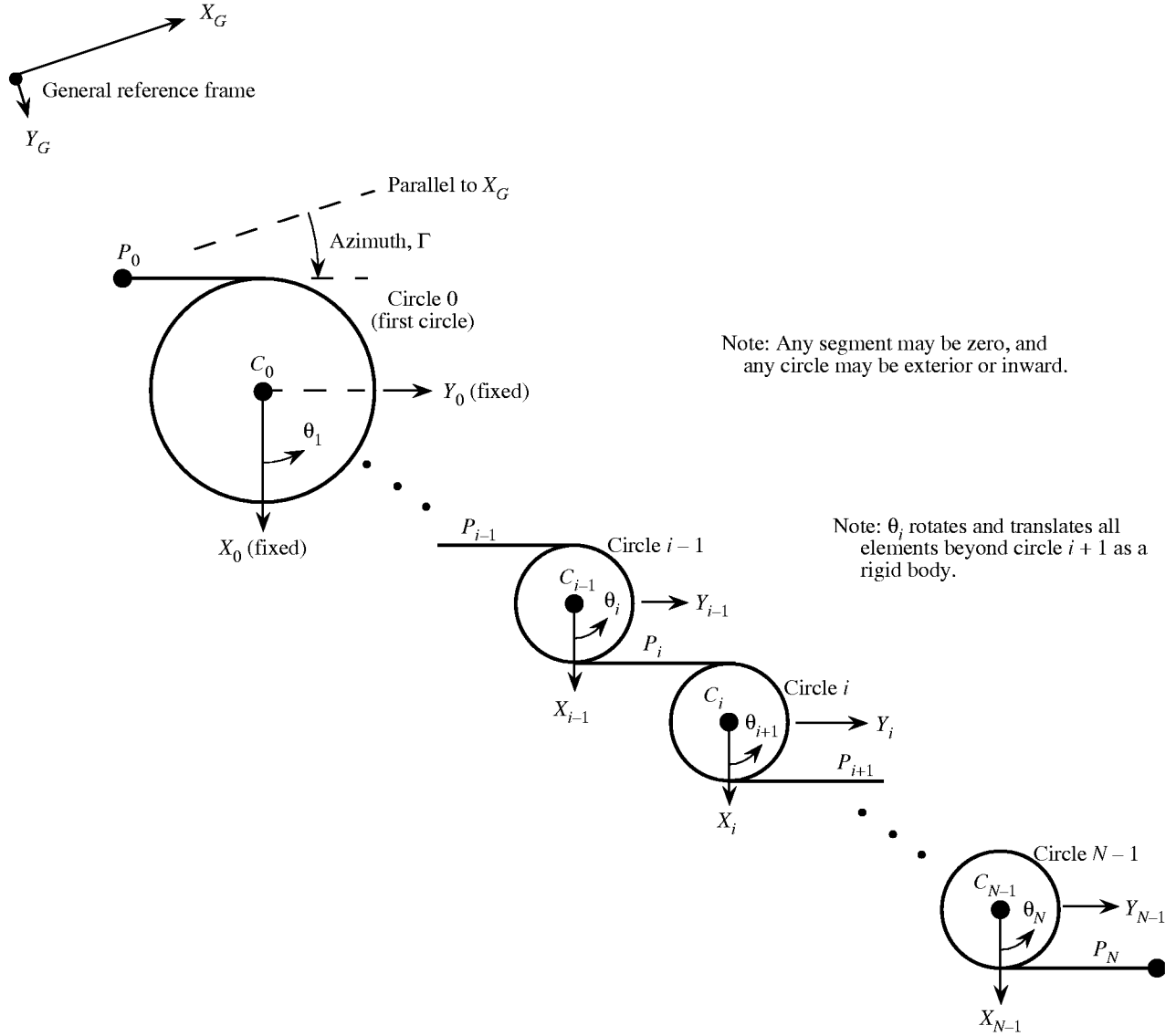
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Figure 3. Zero-reference configurations. All circle angles are 0.

segments are positive, but this initial configuration changes, depending on the type of circle and whether a segment is positive or negative. To change a circle from exterior to inward, simply move the external circle to the opposite side of its upper tangent line; to change a positive segment to a negative segment, simply shift the segment by its length in the opposite direction. As shown, there is a general reference frame (X_G, Y_G, Z_G), plus each circle has its own rigidly attached axis system (at the center of the circle). When $\Gamma = 0$, the axes X_G and Y_0 are aligned. Note that the circle indices begin with zero; that is, the axis system of the first circle is (X_0, Y_0, Z_0). All the axis systems for the circles are parallel when the circles angles are 0. The configuration of

the circles changes to accommodate different path designs.

4.2. Circle Rotation Angle

Think of the center of each circle as a rotational joint and the distance between consecutive circle centers as a link (like in a manipulator robot). The first circle is stationary. A rotation at the center of the first circle (θ_1 about Z_0) translates and rotates the second circle and all remaining circles. If two circles happen to be tangent, the second circle moves along the circumference of the first circle. By convention, the circle angle θ_1 is always about Z_{i-1}. See appendix A

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for an example of configuring a path with circle angles. Later, these circle angles will configure a path but will not be user inputs. (With respect to manipulators, circle angles are analogous to the joint angles.)

4.3. Segment Notation

If there is a segment prior to the first circle C_0, call it P_0. Afterwards, segment P_{i+1} follows circle C_i. Therefore, the last segment P_N follows the last circle C_{N-1}. Hence, there are N circles in a path and up to $N + 1$ linear segments.

5. Smooth Path Design

Figure 4 shows the six possible locations of the second of two immediately consecutive circles in a path. Various paths can be traced (arrows) along the configuration; however, not all are smooth. (A cusp can occur at the transition point between two circles or between a circle and a segment; otherwise, an abrupt change in direction occurs.) Smooth paths (no cusps) are the primary interest in this publication, but the basic analysis is not limited in this respect.

5.1. Direction of Motion on Circle

A counterclockwise (ccw) motion on a circle is positive, and a clockwise (cw) motion is negative.

Symbolically, denote the direction of motion on a circle as

$$\omega_i = \left\{ \begin{array}{ll} 1 & (C_{i-1} \text{ ccw}) \\ -1 & (C_{i-1} \text{ cw}) \end{array} \right\} \qquad (2)$$

5.2. Necessary and Sufficient Conditions for Smooth Path

A computer program can automatically assign information or issue an alert for invalid inputs in the design of a path. For a smooth transition from a circle to a segment, the segment length parameter with algebraic sign is

$$\begin{aligned} l_0 &= \text{Absolute length of segment } P_0 \\ l_i &= \omega_i(\text{Absolute length of segment } P_i) \end{aligned} \right\}$$

$$(i = 1, 2, ..., N) \qquad (3)$$

For a smooth path, another relationship that must hold between immediately successive circles in a path is (fig. 4)

$$\omega_{i+1} = -e_{i+1}\omega_i \qquad (4)$$

where ω_1 determines the direction of motion on the first circle and depends on the initial path direction
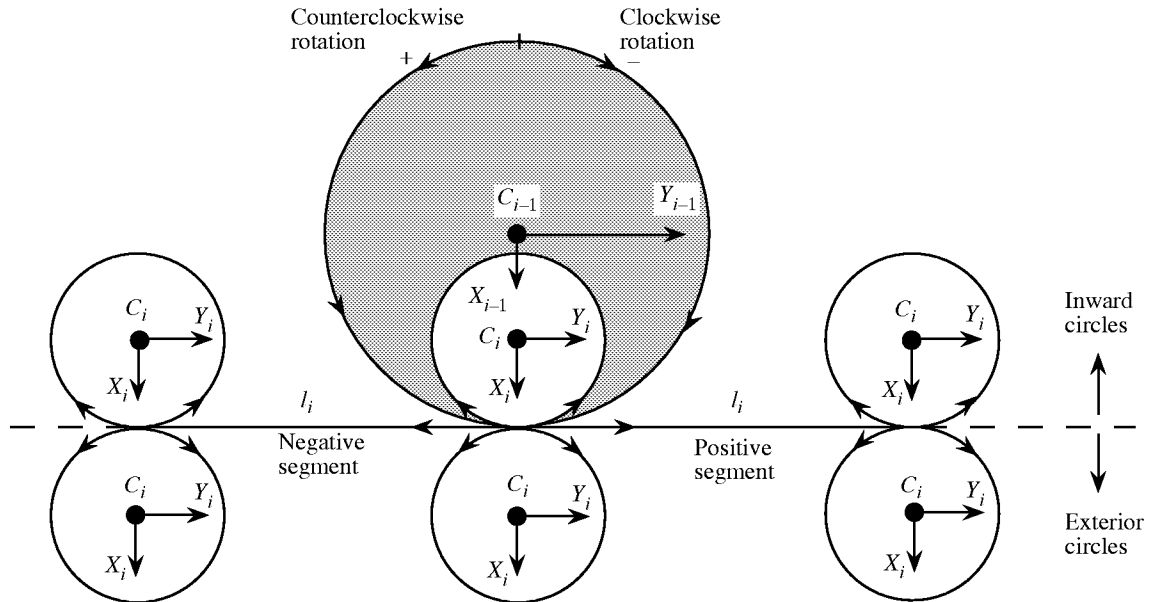

Figure 4. Possible paths (arrows) along next circle.

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(right or left). Equations (3) and (4) constitute necessary and sufficient conditions for a smooth path.

As an example in using equation (4), if the motion on the first of two consecutive circles is clockwise ($\omega_i = -1$) and a counterclockwise motion ($\omega_{i+1} = 1$) is desired on the second circle, then the second circle must be external, as shown by

$$e_{i+1} = -\left(\frac{\omega_{i+1}}{\omega_i}\right) = -\left(\frac{1}{-1}\right) = 1$$

For the same first circle ($\omega_i = -1$), if the second circle were external ($e_{i+1} = 1$), then the direction of motion on the second circle must be counterclockwise, as shown by

$$\omega_{i+1} = -e_{i+1}\omega_i = -(1)(-1) = 1$$

Consequently, beyond the first circle, a user only needs to specify either the circle classification or the direction of motion on the remaining circles.

6. Homogeneous Transformation Matrices Along Path

A homogeneous transformation matrix (ref. 4) completely characterizes the geometric relationship between two axis systems, which may be rotated and/or displaced from each other.

6.1. Transformation Between Consecutive Circles

The homogeneous transformation matrix from circle C_i to the previous circle C_{i-1} (for $i = 1, 2, ..., N$) is

$$A^i_{i-1} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_i\cos\theta_i - l_i\sin\theta_i \\ \sin\theta_i & \cos\theta_i & 0 & a_i\sin\theta_i + l_i\cos\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

where θ_i is the circle angle (positive about Z_{i-1}) from X_{i-1} to X_i, l_i is the distance (either plus or minus) along the tangent segment between the two circles, and

$$a_i = R_{i-1} + e_i R_i \quad (6)$$

is the component of the vector from C_{i-1} to C_i along X_i. For the special case of tangent circles ($l_i = 0$), the magnitude of a_i is the distance between circle centers. The transformation matrix between any two arbitrary circles in a path is a concatenation of the individual transformation matrices between pairs of circles in a path. See appendix B for a brief discussion of homogeneous matrices.

6.2. Transformation From Circle to General Reference Frame

The homogeneous transformation matrix from any circle C_i in the path to the reference frame is

$$A^i_G = A^0_G A^1_0 A^2_1 ... A^i_{i-1} \quad (7)$$

where, for the particular orientation of the general reference frame shown in figure 3,

$$A^0_G = \begin{bmatrix} \cos\Gamma & -\sin\Gamma & 0 & h_0 \\ \sin\Gamma & \cos\Gamma & 0 & k_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

is the transformation from the first circle C_0 and

$$h_0 = x_{G,\text{exit}} - (\text{Exit direction})R_0 \sin\Gamma \quad (9)$$

$$k_0 = y_{G,\text{exit}} + (\text{Exit direction})R_0 \cos\Gamma \quad (10)$$

are the center coordinates of C_0 in the reference frame. Here, $x_{G,\text{exit}}$ and $y_{G,\text{exit}}$ are the coordinates of the exit point (beginning point on first circle in the path). The exit point is where the path first veers away from a reference direction (along runway), and the parameter that categorizes the exit direction is

$$\text{Exit direction} = \left\{\begin{matrix} 1 & (\text{Exiting to right}) \\ -1 & (\text{Exiting to left}) \end{matrix}\right\} \quad (11)$$

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6.3. Recursive Relationship for Circle Center

The center of a circle is important in creating a path and later in following a path. Let $(h_i, k_i, 0)$ be the center point of circle C_i. An equation for the circle center (zero coordinates in circle axis system) is

$$\begin{Bmatrix} h_i \\ k_i \\ 0 \\ 1 \end{Bmatrix} = A_G^{i-1} A_{i-1}^i \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$= A_G^{i-1} \begin{Bmatrix} a_i \cos \theta_i - l_i \sin \theta_i \\ a_i \sin \theta_i + l_i \cos \theta_i \\ 0 \\ 1 \end{Bmatrix} \qquad (12)$$

Or, more explicitly, the following vector recursive relationship is given:

$$\begin{Bmatrix} h_i \\ k_i \\ 0 \end{Bmatrix} = R_G^{i-1} \begin{Bmatrix} a_i \cos \theta_i - l_i \sin \theta_i \\ a_i \sin \theta_i + l_i \cos \theta_i \\ 0 \end{Bmatrix} + P_{i-1}^G$$

$$= \begin{Bmatrix} h_{i-1} \\ k_{i-1} \\ 0 \end{Bmatrix} + R_G^{i-1} \begin{Bmatrix} a_i \cos \theta_i - l_i \sin \theta_i \\ a_i \sin \theta_i + l_i \cos \theta_i \\ 0 \end{Bmatrix}$$

$$(13)$$

To start the recursion,

$$\begin{Bmatrix} h_0 \\ k_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} A_G^0(0,3) \\ A_G^0(1,3) \\ 0 \end{Bmatrix} \qquad (14)$$

or equivalently,

$$\begin{Bmatrix} h_0 \\ k_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} x_{G,\text{exit}} \\ y_{G,\text{exit}} \\ 0 \end{Bmatrix} - \omega_1 \begin{Bmatrix} R_0 \sin \Gamma \\ R_0 \cos \Gamma \\ 0 \end{Bmatrix} \qquad (15)$$

where on the first circle C_0, R_0 is the radius, $(x_{G,\text{exit}}, y_{G,\text{exit}}, 0)$ is the initial point (exit), and ω_1 is the direction of travel.

7. Drawing Path

Several input formats could be used to define a path (user-preference issue). In any event, directly or indirectly from a user's input, the following information about a path is known:

1. Exit point

2. Whether the path is to the right or left

3. The number of circles

4. Whether each circle is exterior or inward

5. Radius of each circle

6. Distance along each linear segment (with direction)

7. Angular travel (travel angle with direction) along each circle

A path can be drawn once a set of design parameters is known. Several reasons for drawing a path are

1. To show users what their design looks like, thereby adding confidence (validity) to the design

2. To provide visual feedback in the design process to changes in parameters

3. To overlay and adjust the path to known geometry (terrain)

4. To provide a graphical reference for motion (visual deviations from a path)

7.1. Path Along Circle

Figure 3 shows the configuration of the circles and segments in a path (for exterior circles and positive segments) when the circle angles are zero. Changing these circle angles from their zero values moves the circles (and segments) to other locations to

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accommodate a particular path design. However, a more natural way to design a path is to specify the angular travel (travel angle) along each circle in the path instead of specifying the circle angles θ_i. (Analogously, in robotics, it is more natural for an operator to command the movements of an end-effector or gripper directly than to specify the joint angles individually in a manipulator.)

7.1.1. Relating Circle Angle to Travel Angle

Define the travel angle λ_i as the angular travel along the circumference of circle C_{i-1} (that is, arc travel along the circumference of a circle subtends the travel angle). A counterclockwise travel angle is positive.
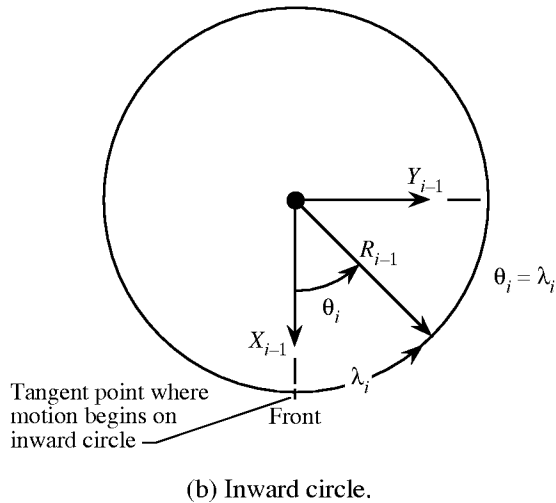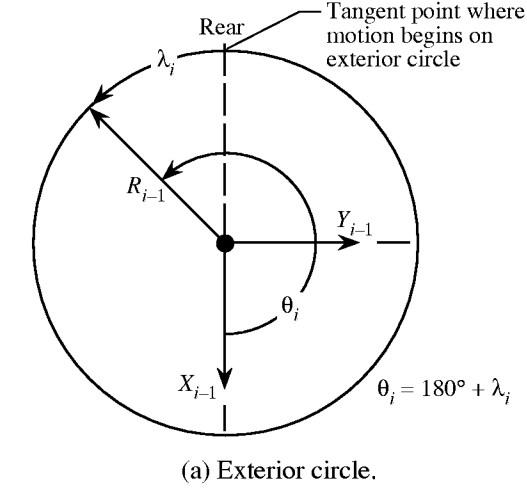

(a) Exterior circle.

(b) Inward circle.

Figure 5. Geometry relating arc travel and circle angle for exterior and inward circles.

Notice in figure 5 that motion begins from the rear (point on circle along $-X_{i-1}$) of an exterior circle and from the front (point on circle along X_{i-1}) of an inward circle. The circle angle θ_i, which moves circle C_i with respect to the circle C_{i-1}, is a function of the travel angle and the circle classification. From figure 5, this relationship is seen to be

$$\theta_i = 90°(1 + e_{i-1}) + \lambda_i \qquad (16)$$

Hence, instead of specifying the circle angles to configure a path, a user specifies the angular travel along each circle and then calculates the circle angles, which is more convenient.

7.1.2. Path Coordinates Along Circle

Let the circle C_i be either an exterior or inward circle that rotates about circle C_{i-1} by the angle λ_{i-1}. The path along circle C_{i-1} is the same whether there is a tangent line from C_{i-1} to C_i. Therefore, for this calculation, assume that the two circles are tangent so that circle C_i touches and rotates along the circumference of circle C_{i-1}. As λ_i varies, a plot of the touch points is the path along circle C_{i-1} (fig. 6). In the reference frame, the path coordinates along circle C_{i-1} are given by

$$\begin{Bmatrix} x_{G,i-1} \\ y_{G,i-1} \\ 0 \\ 1 \end{Bmatrix} = A_G^{i-1} \left[A_{i-1}^i \big|_{l_i = 0} \right] \begin{Bmatrix} -e_i R_i \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

(Path along C_{i-1})

$$(17)$$

where the vector on the right-hand side of equation (17) is the touch point in the axis system of circle C_i and the term in brackets varies with θ_i which varies with λ_i. The segment length l_i in A_{i-1}^i is set equal to 0 in this calculation, which changes at most two elements. Thus, equation (17) gives points along the circular track as an implicit function of the travel angle. To draw the circular track on each circle C_{i-1} (for $i = 1, 2, ..., N$), vary the travel angle λ_i from 0 to its final value. If desired, equation (17)
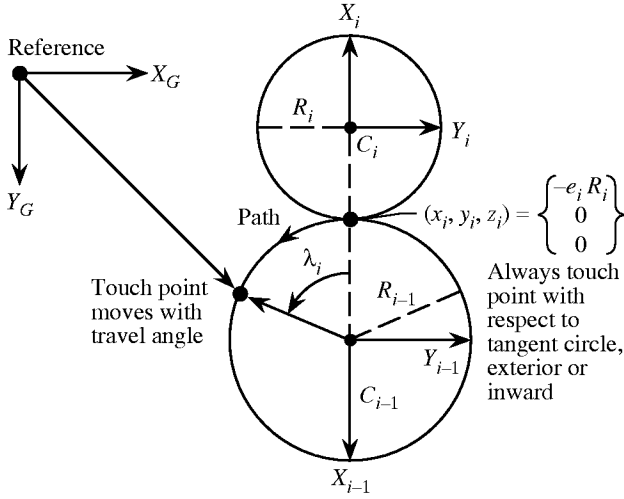
Figure 6. Coordinates on circular path which depend on travel angle.

expands to the more explicit vector expression as follows:

$$\left\{ \begin{array}{c} x_{G,i-1} \\ y_{G,i-1} \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} h_{i-1} \\ k_{i-1} \\ 0 \end{array} \right\} + R_G^{i-1}$$

$$\times \left\{ \begin{array}{c} -e_i R_i \cos \theta_i + a_i \cos \theta_i \\ -e_i R_i \sin \theta_i + a_i \sin \theta_i \\ 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} h_{i-1} \\ k_{i-1} \\ 0 \end{array} \right\} + R_G^{i-1} \left\{ \begin{array}{c} R_{i-1} \cos \theta_i \\ R_{i-1} \sin \theta_i \\ 0 \end{array} \right\} \quad (18)$$

which means that a vector from the reference frame to a path point on circle C_{i-1} is equal to the vector from the reference frame to the center of circle C_{i-1} plus the vector from the center of circle C_{i-1} to the path point (where the matrix multiplication transforms the components of the latter vector to the reference frame). The initial point on a circular track is where the travel angle is 0, and the final point is where the travel angle takes on its final value. If the segment length between two circles is 0 then the ending point on the first circle is the starting point on the second circle. (See appendix C for specific equations to compute the beginning or ending points on any circle in a path.)

In essence, equation (17) uses the next circle to draw a path on the current circle. Consequently, a fictitious circle is required to draw the path along the last circle C_{i-1}. For $i = N$, equation (17) should give the path along the last circle C_{N-1}, but up to now, there is no matrix A_{N-1}^N because there is no circle C_N. To use equation (17) to draw the path along the last circle C_{N-1}, think of a fictitious circle C_N with arbitrary radius R_N at the end of the final segment P_N. The homogeneous transformation for this fictitious circle is

$$A_{N-1}^N = \begin{bmatrix} \cos \theta_N & -\sin \theta_N & 0 & a_N \cos \theta_N - l_N \sin \theta_N \\ \sin \theta_N & \cos \theta_N & 0 & a_N \sin \theta_N + l_N \cos \theta_N \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(19)$$

where

$$\left. \begin{array}{l} a_N = R_{N-1} + e_N R_N \\ \theta_N = 90^\circ(1 + e_{N-1}) + \lambda_N \end{array} \right\} \quad (20)$$

Here, λ_N is the angular travel on circle C_{N-1}, and θ_N is the angular rotation angle of circle C_N about circle C_{N-1}. Computation of this transformation is transparent and automatic with respect to a user.

7.2. Path Along Segment

A straight-line segment can occur

1. Prior to the first circle

2. Between two circles

3. After the last circle

In the first case, the initial point is known. Connecting this initial point and the exit point (which is the starting point on the first circle C_0) plots the first segment.

If there is a segment between two consecutive circles in a path, this segment connects the ending point of the first circle with the starting point of the second circle. Therefore, automatically, the segments will be drawn when the circle path points are connected.

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Lastly, connecting the final path point on the last circle C_{N-1} to the last point in a path plots the final segment P_N. If there is a final segment P_N beyond the last circle, then the coordinates of the last point on this segment are the coordinates of the center of the fictitious circle C_N when $R_N = 0$. Equation (13) gives the center of the fictitious circle C_N as

$$\begin{Bmatrix} h_N \\ k_N \\ 0 \end{Bmatrix} = \begin{Bmatrix} h_{N-1} \\ k_{N-1} \\ 0 \end{Bmatrix} + R_G^{N-1} \begin{Bmatrix} a_N \cos \theta_N - l_N \sin \theta_N \\ a_N \sin \theta_N + l_N \cos \theta_N \\ 0 \end{Bmatrix}$$

(21)

Temporarily, set $R_N = 0$. This means that the tangent point of the segment P_N with the fictitious circle C_N is the same as the center of the fictitious circle. Also, this makes $a_N = R_{N-1}$; additionally, let $e_N = -1$ so that $\theta_N = \lambda_N$. Then, the last point on P_N is given by

$$\begin{aligned} (P_N)_{\text{last point}} = {} & \begin{Bmatrix} h_{N-1} \\ k_{N-1} \\ 0 \end{Bmatrix} + R_G^{N-1} \\ & \times \begin{Bmatrix} R_{N-1} \cos \lambda_N - l_N \sin \lambda_N \\ R_{N-1} \sin \lambda_N + l_N \cos \lambda_N \\ 0 \end{Bmatrix} \end{aligned}$$

(22)

Geometrically, equation (22) says that a vector from the reference frame to the last point in a path is the vector sum of a vector from the origin of the reference frame to the center of the last circle plus a vector (transformed to the reference frame) from the center of the fictitious last circle to the last point on the path. (See appendix C for specific equations to compute the beginning or ending point on any segment in a path.)

7.3. Other Patterns Not Necessarily Smooth

Smooth paths are the primary interest in this publication, but the analysis applies to the design of a variety of patterns not just those that are smooth; see examples in appendix D.

8. Applications in Path Design

A computer program was written (in the C programming language) to compute some example paths. Two example runway exit paths are given for illustration: (1) the first path is along two exterior tangent circles and (2) the second approximates a spiral-type path.

For simplicity, assume that the runway (think of P_0 in fig. 3) is parallel to the reference axis X_G and points north (that is, $\Gamma = \psi_{x_G} = 0$). Let the exit path be to the right (relative to motion just prior to the first circle). Although only examples of runway exit paths are given, the analysis applies equally well to a runway entrance and takeoff path.

8.1. Example 1: Runway Exit Path Using Two Exterior Tangent Circles

In reference 2, an airplane runway exit geometry has an initial constant radius of 2900 ft which makes a transition tangentially, after traveling 30°, into a final constant radius of 1500 ft. The travel angle on the second circle is also 30°. The final point on the exit path is approximately 2200 ft longitudinally and 600 ft laterally from the start of the exit turn.

Example 1 is a path along two tangent circles (both exterior). Because the exit path is to the right, the direction of travel on the first circle is clockwise about z_0 and the direction of travel on the second circle is counterclockwise about z_1. The appropriate parameters to generate the path are as follows:

Two circles, N 2
Radius of first circle, R_0 2900
Radius of second circle, R_1 1500
First circle is exterior, e_0 1
Second circle is also exterior, e_1 1
Angular travel on first circle, λ_1, deg .... −30
Angular travel on second circle, λ_2, deg 30

Assume that the exit turn starts at 3000 ft on the runway. The resulting path, which agrees with that in reference 2, is shown in figure 7. (For a left exit, set $e_0 = -1$ and interchange the travel angles.)
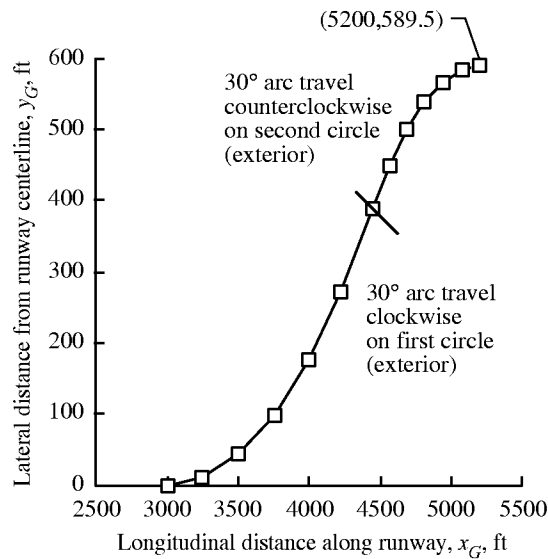
10

Figure 7. Example 1: two-circle runway exit path.

8.2. Example 2: Approximating Spiral-Type Path

Example 2 approximates the spiral-type path in reference 3. This approximation uses two circles at the beginning and at the end of the path with a linear segment in between. The first circle in each pair is exterior and the second is inward. The attraction of a spiral path is that it provides a smooth transition from the runway to the path. An airplane using a runway exit path that is initially spiral (decreasing radius) has a gradual buildup in the lateral acceleration for passenger comfort. Reference 3 uses 26 points (connected with linear segments) to approximate such a path to within a 2-ft error. Here, however, path parameters were varied until there was a reasonable match between the approximate and the ideal paths. No attempt was made to design the path to within any particular error criterion. The basic type of information is as follows (a user need not be aware of the particular variable symbol):

Number of circles in path, N 4

Exit direction, Exit direction $= 1$ .... Right

Circle radius, ft, for —
First circle, R_0 9100
Second circle, R_1 1700
Third circle, R_2 1500
Fourth circle, R_3 800

Direction of motion on —
First circle, $\omega_1 = -1$ .... Clockwise
Second circle, $\omega_2 = -1$ .... Clockwise
Third circle, $\omega_3 = 1$ .... Counterclockwise
Fourth circle, $\omega_4 = 1$ .... Counterclockwise

Angular travel, deg, on —
First circle, $\lambda_1 = -4°$ 4
Second circle, $\lambda_2 = -26°$ 26
Third circle, $\lambda_3 = 8°$ 8
Fourth circle, $\lambda_4 = 21°$ 21

Circle classification:
First circle, $e_0 = 1$ .... Exterior
Second circle, $e_1 = -1$ .... Inward
Third circle, $e_2 = 1$ .... Exterior
Fourth circle, $e_3 = -1$ .... Inward

Segment length, ft, for —
Initial segment (in front of first circle), l_0 ..0
Between first and second circle, l_1 0
Between second and third circle, l_2 416
Between third and fourth circle, l_3 0
After last (fourth) circle, l_4 0

The approximated path is shown in figure 8 and agrees well with the exact path.

9. Following a Path

A path is nothing more than a geometric curve (road). In trying to follow a path, one will deviate from the path; therefore, a correction is needed. A method to follow a path (refs. 1 and 2) is to orthogonally project the current position (moving point in reference frame) onto the path as a target point and then make a correction to move toward this target point to reduce the lateral deviation from the path. (Think of tracking your shadow on the path which is always directly below or above you.) This approach appears successful in controlling lateral deviation and is the motivation for this section. However, actually controlling motion to follow a path (refs. 1, 2, and 3) is not part of this publication.

9.1. Target Point on Circle

Imagine a new axis system (X'_i, Y'_i, Z'_i) at the center of circle C_i that is always parallel to the general reference frame (X_G, Y_G, Z_G) as shown in figure 9. Let $(x_G, y_G, 0)$ be a known, but arbitrary, point in the
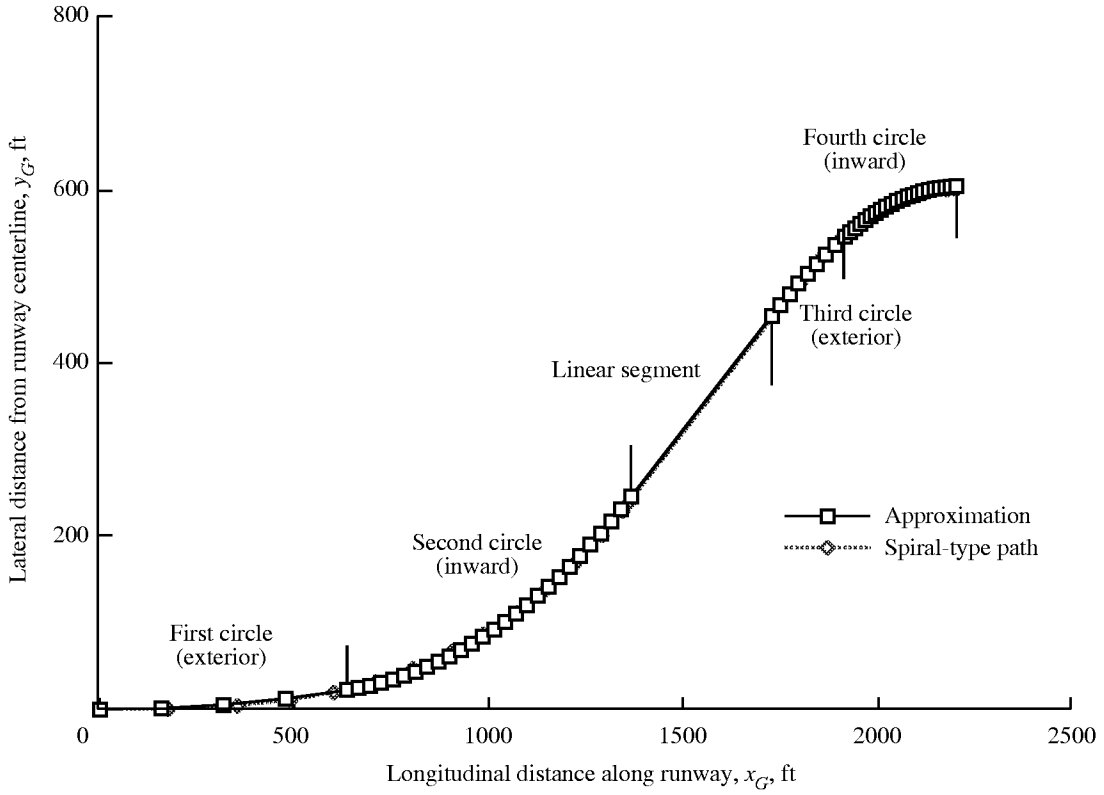
11

Figure 8. Example 2: spiral-type runway exit path.

general reference frame and project this point along the radius onto the circle. Refer to this point on the circle as the target point.

To compute the target point, first define a projection angle with respect to the (X'_i, Y'_i, Z'_i) axis system as

Figure 9. Radial projection of known point to circle.

$$\phi'_i = \tan^{-1}\left(\frac{x_G - h_i}{y_G - k_i}\right) \qquad (-180° < \phi'_i \le 180°) \quad (23)$$

The target point lies along the same radial line, from the center of the circle, as the known point, which means that the two points have the same projection angle. Therefore, in this axis system, the target-point coordinates are

$$(x'_i)_{\text{target}} = R_i \sin \phi'_i \qquad (24)$$

$$(y'_i)_{\text{target}} = R_i \cos \phi'_i \qquad (25)$$

and, in general reference frame coordinates, the target point coordinates are

$$(x_{G,i})_{\text{target}} = h_i + (x'_i)_{\text{target}} \qquad (26)$$

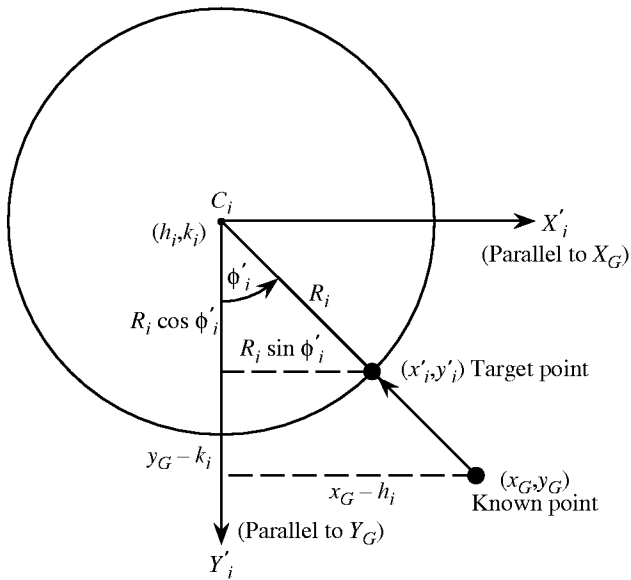$$(y_{G,i})_{\text{target}} = k_i + (y'_i)_{\text{target}} \qquad (27)$$

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9.2. Target Point on Segment

The starting and ending points on a segment define a line. This section gives a set of equations to compute a target point on this line. In general, if two straight lines intersect at right angles, the intersection point is the orthogonal projection of any point on either line onto the other line. Let $(x_{G,1}, y_{G,1}, 0)$ and $(x_{G,2}, y_{G,2}, 0)$ be two points that define a line L in the general reference frame (for example, the two points may be the beginning and ending points of a segment). Let $L_\perp$ be another line that (1) is perpendicular to L and (2) passes through an arbitrary but known point $(x_{G,a}, y_{G,a}, 0)$. The intersection of the two lines defines the orthogonal projection of the known point. The orthogonal projection is the target point. Obviously, if L is parallel to X_G, the target point is

$$\left\{ \begin{array}{c} x_G \\ y_G \\ 0 \end{array} \right\}_{\text{target}} = \left\{ \begin{array}{c} x_{G,1} \\ y_G \\ 0 \end{array} \right\} \quad \begin{array}{l} (x_{G,1} = x_{G,2}; \\ y_G = y_{G,a}) \end{array} \tag{28}$$

Otherwise, the target point is the intersection of the two lines:

$$y_G = m x_G + (y_{G,1} - m y_{G,1}) \tag{29}$$

$$y_G = -\frac{1}{m} x_G + \left(y_{G,a} + \frac{1}{m} x_{G,a} \right) \tag{30}$$

Call the intersection point the target point and make the notational change $(x_G, y_G) = (x_{G,a}, y_{G,a})$ to get

$$\left\{ \begin{array}{c} x_G \\ y_G \\ 0 \end{array} \right\}_{\text{target}} = \frac{1}{1 + m^2}$$

$$\times \left\{ \begin{array}{c} m(y_G - y_{G,1}) + x_G + m^2 x_{G,1} \\ m(x_G + x_{G,1}) + y_{G,1} + m^2 y_G \\ 0 \end{array} \right\}$$

$$(x_{G,1} \neq x_{G,2}) \tag{31}$$

where

$$m = \frac{y_{G,2} - y_{G,1}}{x_{G,2} - x_{G,1}} \tag{32}$$

is the slope of the known line or segment and (x_G, y_G) now denotes the current position.

9.3. Target Azimuth on Circle

The azimuth of a moving target point on a circle is useful in correcting the heading of current motion. The azimuth at an arbitrary point on circle C_i is

$$\psi_i = \psi_{X_G} + \psi_{G,i} \qquad (i = 0, 1, ..., N) \tag{33}$$

where ψ_{X_G} is the known azimuth of the axis X_G and $\psi_{G,i}$ is the azimuth of the moving point relative to X_G. Next consider how $\psi_{G,i}$ varies.

The azimuth on circle C_i with respect to X_G is shown in figure 10(a) as a function of the projection angle ϕ_i' and the direction of motion on the circle. For example, if $\phi_i' = 90°$, figure 10(a) shows an azimuth of $\psi_{G,i} = 90°$ for a clockwise motion, whereas figure 10(b) shows an azimuth of $\psi_{G,i} = 270°$ for a counterclockwise motion.

Figure 10(c) shows a plot of $\psi_{G,i}$ as a function of ϕ_i' and the direction of motion. Equations for computer computations are

$$\left. \begin{array}{ll} \psi_{G,i} = 180° - \phi_i' & (C_i \ \text{ccw}; -180° \leq \phi_i' \leq 180°) \\ \psi_{G,i} = -\phi_i' & (C_i \ \text{cw}; \phi_i' \leq 0°) \\ \psi_{G,i} = 360° - \phi_i' & (C_i \ \text{cw}; \phi_i' > 0°) \end{array} \right\} \tag{34}$$
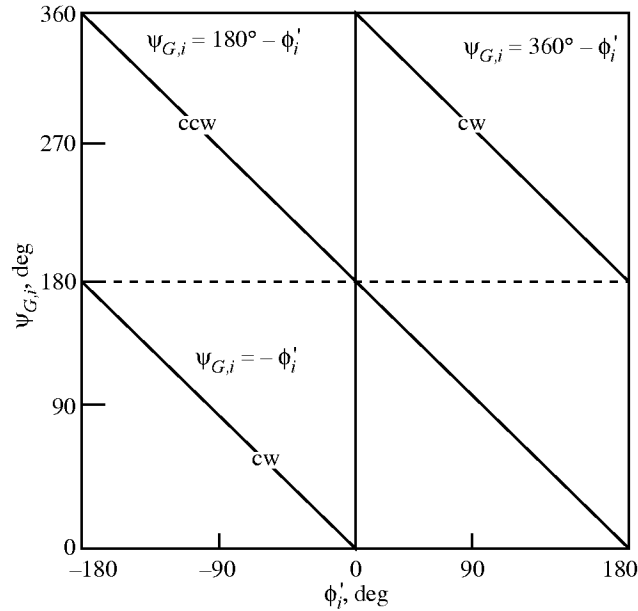
where ϕ_i' is given by equation (23). The scenario is that current motion near a circle changes the circle projection angle ϕ_i' in equation (23), and this changes the azimuth in equations (34), which changes the target azimuth in equation (33).
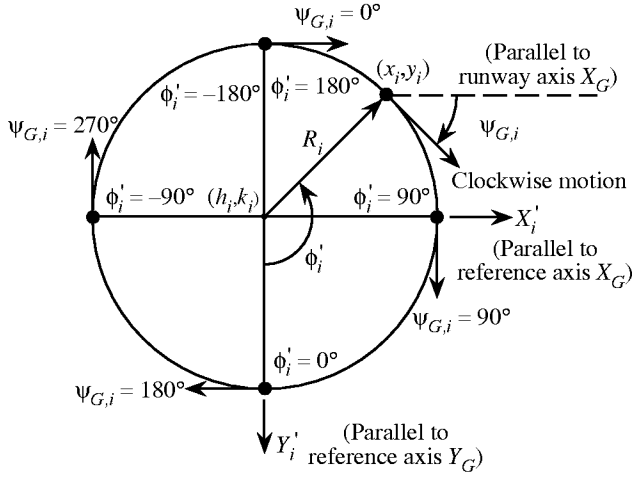
(a) Clockwise motion on circle.
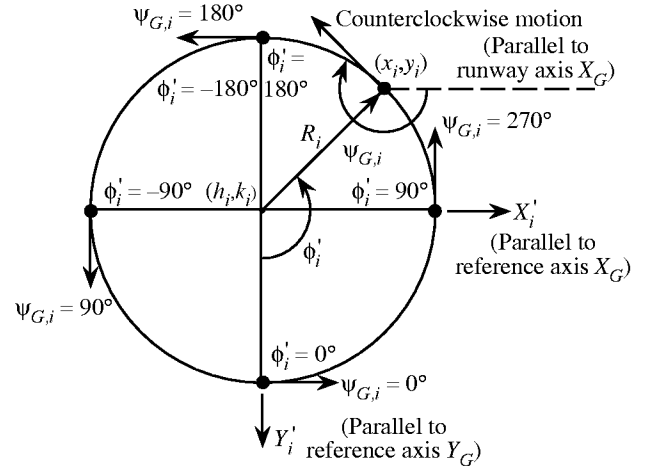
(b) Counterclockwise motion on circle.

(c) Azimuth angle as function of circle projection angle and direction of rotation on circle.

Figure 10. Circle azimuth.

9.4. Target Azimuth on Segment

The constant target azimuth on a segment P_i is

$$\psi_{P_i} = \psi_{X_G} + \tan^{-1}\left(\frac{y_{G,2} - y_{G,1}}{x_{G,2} - x_{G,1}}\right) \qquad (35)$$

where $(x_{G,1}, y_{G,1})$ is the starting point on the segment and $(x_{G,2}, y_{G,2})$ is the ending point.

Alternatively, the azimuth on a segment prior to the first circle is $\psi_{P_0} = \psi_G + \Gamma$; the azimuth angle on any other segment following a circle is the same as the last azimuth angle on that circle (that is, the azimuth at the ending point of the circle).

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10. Switching Motion to Next Path Element

10.1. Sample Flowchart for Referencing Motion to Path

Figure 11 shows a flowchart to reference motion to a path. The diagram sequences through the path along its various segments and circles. Assume there are N circles in a path, along with a final fictitious circle (for convenience). Then, for each path index $(i = 0, 1, ..., N)$, there is a segment followed by a circle. The fictitious circle follows the last segment. Initially, the first element is P_0, but if the length of this segment is zero, the first circle C_0 then becomes the reference for the current motion. After completing motion on C_0, the loop index increases by one to the
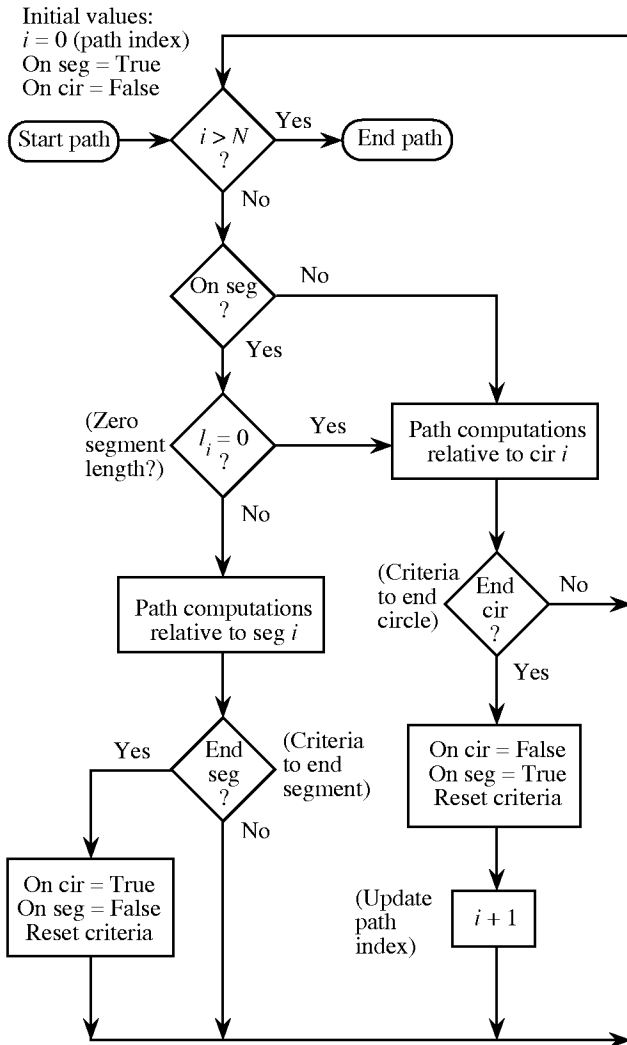
next segment in the path P_1 and so forth. The loop continues until there are no more elements in the path.

10.2. Switching Motion to Next Path Element

A user supplies some criteria to decide when to switch from one path element to the next. Numerous possible switching criteria (a research issue) may exist; for example, each element in the path has its own angle to go before the next element starts. When this angle reaches 0, motion switches to the next element and so on.

This section defines an angle to go to the next successive element in a path. This switching angle is the central angle between the currently known position, a circle center, and the start (or end) of motion on the circle. Figure 12 indicates the switching angle for motion approaching an exterior circle. Bear in mind that the switching angle is only one indication of when to switch to the next path element; it does not indicate how close the motion is to the path at the instant of switching. Other parameters, such as the lateral position from the path, indicate the "closeness" of the motion to the path and also play an integral role in the switching decision.

10.2.1. Switching Motion Away From Segment

First, assume that motion is currently relative to the segment P_i. Then, the angle to go tells when to switch motion to the upcoming circle C_i. Let $(x_G, y_G, 0)$ be a known moving point in the general reference frame. A transformation of this point to the circle axis system is

$$\begin{Bmatrix} x_i \\ y_i \\ 0 \\ 1 \end{Bmatrix} = A_i^G \begin{Bmatrix} x_G \\ y_G \\ 0 \\ 1 \end{Bmatrix} \qquad (36)$$

The radial projection angle of the moving point in the circle axis system is then

$$\phi_i = \tan^{-1}\left(\frac{y_i}{x_i}\right) \qquad (-180° < \phi_i \le 180°) \quad (37)$$

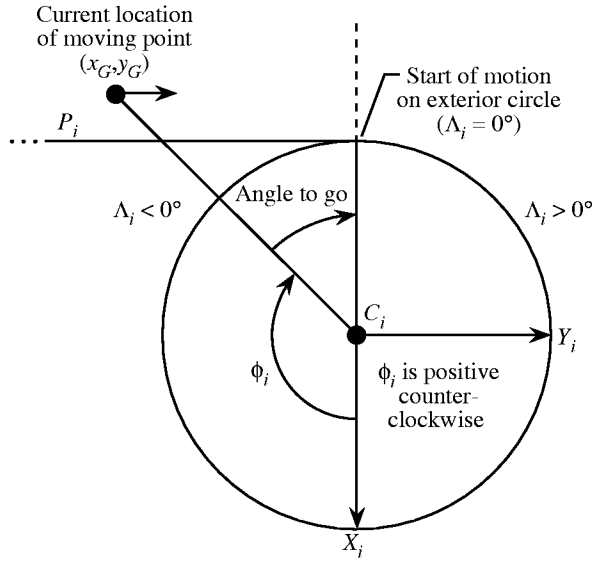Figure 11. Sequencing motion along path elements.

Figure 12. Angular location of current position from starting point on exterior circle.

The central angle of travel before motion starts on circle C_i is

$$\Lambda_i = -(\text{sign of } \omega_{i+1}\phi_i)(1 + e_i)90° + \omega_{i+1}\phi_i \quad (38)$$

which is a function of the current projection angle, circle classification, and the direction of motion. Equation (38) results by considering the various possibilities of approaching the start of motion on a circle. In essence,

$$\Lambda_i < 0 \quad (\text{Before motion starts on } C_i)$$
$$\Lambda_i = 0 \quad (\text{Start motion on } C_i)$$
$$\Lambda_i > 0 \quad (\text{Motion on } C_i)$$

(If there is a segment P_N following the last circle in a path, a user can conveniently choose to set $R_N = |l_N|$ to make Λ_N vary from $-45°$ to $0°$ as motion starts and stops on this last segment.) Define an angle to go that is positive before motion is to start on C_i as

$$\text{Angle to go} = -\Lambda_i \quad (39)$$

10.2.2. Switching Motion Away From Circle

If motion is currently on circle C_i, then the angle to go should indicate when to switch motion away from C_i onto the next upcoming path element (circle

or segment). In this case the angle to go is to the end of motion on C_i. Although motion is on C_i, Λ_i in equation (38) is positive and gives the central angle of travel on C_i. Because the desired angular travel on C_i is known, the angle to go for switching motion away from circle C_i is

$$\text{Angle to go} = \left|(\lambda_i)_{\max}\right| - \Lambda_i \quad (40)$$

Normally, one knows the current position and then computes the angle to go to switch to the next element in a path.

11. Comments on Control

Knowing the angle to go and how this angle is currently changing with time, one can compute a time to go to switch to the next path element and, if desired, can start the switching maneuver a fraction of a second early for a better transition (less overshoot). In dealing with straight-line segments, reference 1 divides the distance to go by the current speed to get a time to go to the end of the segment. When the time to go is within 0.05 sec, tracking switches to the next segment in the path.

At a switching point in a path, there is a call for an instantaneous change in a velocity component; for example, in exiting from a runway onto a circular path element, there is a call for the velocity normal to the runway to change immediately from 0 to a value that is consistent with motion in a circle. Reference 2 starts to build the normal velocity component slightly before reaching the exit for a smoother transition off the runway, and reference 3 uses a spiral-type exit for smoothness. Error equations for controlling motion to follow a path are given in reference 1.

Reference 5 presents an algorithm, although not used, for lateral ride comfort limits. Current thinking (ref. 3) is that a lateral acceleration should not exceed $0.15g$ and that lateral jerk (derivative of acceleration) should not be in excess of 0.05 g/sec.

12. Concluding Remarks

This publication provides a general framework for the mathematical description of a smooth path along an arbitrary combination of circles and tangent lines.

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The innovative approach is to model the path as a giant multilink manipulator and apply the mathematics common to robotics. Any two successive circles in a path are either tangent or connected by a common tangent line. There may be an initial segment prior to the first circle in the path, and a path may begin or end on a circle or line. A user specifies a few parameters to configure the path. Necessary and sufficient conditions ensure that user inputs are consistent with a smooth path. An immediate application is designing smooth paths for use in automating the exit of an aircraft from a runway. Two example runway exit paths are given, and this publication defines an angle to go for switching from one path element to the next.

Appendix A

Configuring Path With Circle Angles

   Appendix A illustrates geometrically how an initial configuration of circles and tangent lines changes with circle angles in the design of a smooth path. However, in the main text circle angles are not explicit inputs in the design process.

   Figure A1 shows an example of configuring a path with circles and tangent lines. In figure A1(a), the initial configuration, prior to designing a path, is when the circle angles are zero. The configuration then changes with circle angles to design a path. The first two circles are exterior (the first by definition), and the third is an inward circle. There is a tangent segment prior to the first circle and another tangent segment between the second and third circles. Figure A1(b) shows the configuration after a rotation of $\theta_1 = 110°$ about the center of the first circle, and figure A1(c) shows the final configuration after a rotation of $\theta_2 = -110°$ about the center of the second circle. The example smooth path is then drawn along the circles and segments. The amount of travel along the last circle is $\theta_3 = 60°$.
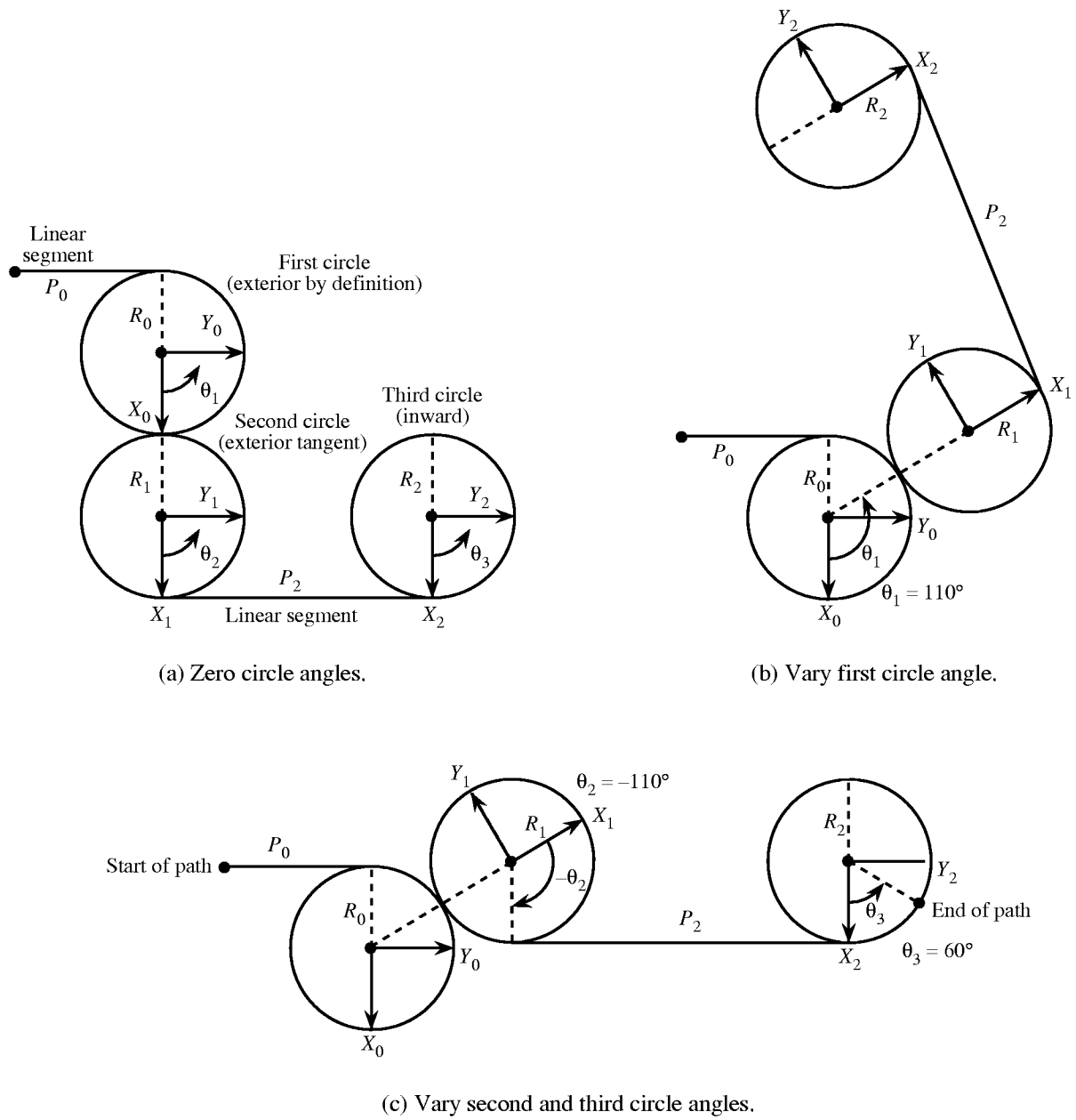
(a) Zero circle angles.

(b) Vary first circle angle.

(c) Vary second and third circle angles.

Figure A1. Constructing smooth path along circular arcs and tangent segments using circle angles.

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Appendix B

Homogeneous Transformation Matrices

Homogeneous transformation matrices contain both orientation and displacement information between axis systems and are useful in following the path geometry. See reference 4 for application of homogeneous transformation matrices to robotics. Consider the two axis systems a and b shown in sketch B1:
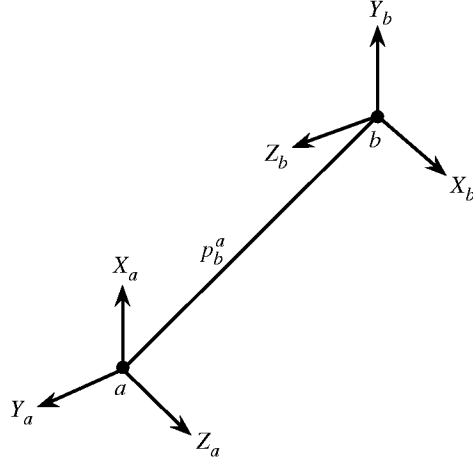
Sketch B1

The two axis systems are related by the following 4×4 homogeneous transformation matrix:

$$A_a^b = \left[\begin{array}{ccc:c} & R_a^b & & p_b^a \\ \hdashline 0 & 0 & 0 & 1 \end{array} \right] \tag{B1}$$

where R_a^b is a 3×3 rotational matrix that transforms a three-dimensional vector in axis system b to axis system a:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = R_a^b \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{B2}$$

and

$$p_b^a = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}_a \tag{B3}$$

is a 3×1 position vector directed from the origin of the axis system a to the origin of axis system b.

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B1. Inverse Transformation

The homogeneous transformation matrix to transform quantities from axis system a to axis system b is the inverse of equation (B1):

$$A_b^a = \left[A_a^b\right]^{-1} = \left[\begin{array}{ccc|c} & \left[R_b^a\right]^T & & -\left[R_b^a\right]^T p_b^a \\ \hline 0\ 0\ 0 & & & 1 \end{array}\right] \tag{B4}$$

where the superscript T represents the transpose of the matrix.

B2. Transforming Points and Vectors

A homogeneous transformation matrix transforms the coordinates of a point and the components of a vector as follows:

$$\left\{\begin{array}{c} x \\ y \\ z \\ 1 \end{array}\right\}_{i-1} = A_{i-1}^i \left\{\begin{array}{c} x \\ y \\ z \\ 1 \end{array}\right\}_i \qquad \text{(Transforming coordinates of point)} \tag{B5}$$

$$\left\{\begin{array}{c} v_x \\ v_y \\ v_z \\ 0 \end{array}\right\}_{i-1} = A_{i-1}^i \left\{\begin{array}{c} v_x \\ v_y \\ v_z \\ 0 \end{array}\right\}_i \qquad \text{(Transforming vector, such as velocity)} \tag{B6}$$

B3. Specific Partitioning (Circle-to-Circle Matrix)

The homogeneous transformation matrix from circle C_i to the previous circle C_{i-1} is

$$A_{i-1}^i = \left[\begin{array}{ccc|c} & R_{i-1}^i & & p_i^{i-1} \\ \hline 0\ 0\ 0 & & & 1 \end{array}\right] \tag{B7}$$

where

$$R_{i-1}^i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B8}$$

is an inverse Euler rotation matrix about Z_{i-1}, and

$$p_i^{i-1} = \begin{Bmatrix} a_i \cos \theta_i - l_i \sin \theta_i \\ a_i \sin \theta_i + l_i \cos \theta_i \\ 0 \end{Bmatrix} \tag{B9}$$

is the position vector from the center of circle C_{i-1} to the center of circle C_i (p_i^{i-1} has components in $(X_{i-1}, Y_{i-1}, Z_{i-1})$).

Appendix C

Path Elements at Starting and Ending Points

In switching from one path element to the next, one needs to know where the current element ends and the next begins. Appendix C gives equations for calculating the beginning and ending points on all circles and segments in a path.

For a given path configuration, the homogeneous transformation matrix A_G^i from circle C_i to the reference frame is constant. In addition, for later convenience, define the matrix

$$L_G^{i+1} \equiv A_G^i \left[A_i^{i+1} \right]_{l_{i+1}=0} \tag{C1}$$

which is the same as matrix A_G^{i+1}, except that l_{i+1} is 0 in A_i^{i+1}; that is, except for the two elements:

$$\left. \begin{aligned} L_G^{i+1}(0,3) &= a_{i+1} \cos \theta_{i+1} \\ L_G^{i+1}(1,3) &= a_{i+1} \sin \theta_{i+1} \end{aligned} \right\} \tag{C2}$$

C1. Circles

The starting point of motion on circle C_i is

$$\begin{Bmatrix} x_{G,i}^* \\ y_{G,i}^* \\ 0 \\ 1 \end{Bmatrix} = A_G^i \begin{Bmatrix} -e_i R_i \\ 0 \\ 0 \\ 1 \end{Bmatrix} \qquad (i = 0, 1, ..., N) \tag{C3}$$

where the superscript * identifies a starting point. If desired, equation (C3) expands to the more explicit vector expression:

$$\begin{Bmatrix} x_{G,i}^* \\ y_{G,i}^* \\ 0 \end{Bmatrix} = \begin{Bmatrix} h_{i-1} \\ k_{i-1} \\ 0 \end{Bmatrix} + R_G^{i-1} \begin{Bmatrix} -e_i R_i \cos \theta_i + a_i \cos \theta_i - l_i \sin \theta_i \\ -e_i R_i \sin \theta_i + a_i \sin \theta_i + l_i \cos \theta_i \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} h_{i-1} \\ k_{i-1} \\ 0 \end{Bmatrix} + R_G^{i-1} \begin{Bmatrix} R_{i-1} \cos \theta_i - l_i \sin \theta_i \\ R_{i-1} \sin \theta_i + l_i \cos \theta_i \\ 0 \end{Bmatrix} \tag{C4}$$

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On the imaginary circle C_N, the starting point is the same as the ending point because there is no motion on this circle; that is

$$
\begin{Bmatrix} x^{\#}_{G,N} \\ y^{\#}_{G,N} \\ 0 \end{Bmatrix} = \begin{Bmatrix} x^{*}_{G,N} \\ y^{*}_{G,N} \\ 0 \end{Bmatrix} \tag{C5}
$$

where the superscript # identifies the ending point. The ending points for all other circles are given by

$$
\begin{Bmatrix} x^{\#}_{G,i} \\ y^{\#}_{G,i} \\ 0 \\ 1 \end{Bmatrix} = L^{i+1}_{G} \begin{Bmatrix} -e_{i+1} R_{i+1} \\ 0 \\ 0 \\ 1 \end{Bmatrix} \qquad (i = 0, 1, ..., N-1) \tag{C6}
$$

If desired, equation (C6) readily expands to

$$
\begin{Bmatrix} x^{\#}_{G,i} \\ y^{\#}_{G,i} \\ 0 \\ 1 \end{Bmatrix} = A^{i}_{G} \begin{Bmatrix} R_i \cos \theta_{i+1} \\ R_i \sin \theta_{i+1} \\ 0 \\ 1 \end{Bmatrix} \qquad (i = 0, 1, ..., N-1) \tag{C7}
$$

C2. Segments

The initial point on any initial segment prior to the first circle in a path is either known or computable from the exit coordinates, segment length, and azimuth angle. The final point on such a segment is the known exit point (or the starting point on the first circle). Hence,

$$
(P_0)_{\text{start}} = \text{Known or computable point} \tag{C8}
$$

$$
(P_0)_{\text{end}} = \text{Exit point (known)} \tag{C9}
$$

Thereafter, the starting point on segment P_i is the ending point on the preceding circle:

$$
(P_i)_{\text{start}} = (C_{i-1})_{\text{end}} = \begin{Bmatrix} x^{\#}_{G,i-1} \\ y^{\#}_{G,i-1} \\ z^{\#}_{G,i-1} \end{Bmatrix} \qquad (i = 1, 2, ..., N) \tag{C10}
$$

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The ending point on P_i is the starting point on the upcoming circle:

$$(P_i)_{\text{end}} = (C_i)_{\text{start}} = \left\{ \begin{array}{c} x^*_{G,i} \\ y^*_{G,i} \\ z^*_{G,i} \end{array} \right\} \qquad (i = 1, 2, ..., N) \qquad \text{(C11)}$$

As a matter of interest, after the first segment P_0, the relationship between the end point of segment P_i (which is also the relationship between the ending point on C_{i-1} and the starting point on C_i) is

$$(P_i)_{\text{start}} = (P_i)_{\text{end}} - R_G^{i-1} \left\{ \begin{array}{c} -l_i \sin \theta_i \\ l_i \cos \theta_i \\ 0 \end{array} \right\} \qquad (i = 1, 2, ..., N) \qquad \text{(C12)}$$

Appendix D

Interesting Patterns, Not Necessarily Smooth

One can generate some interesting patterns using circles and lines. Not necessarily designing patterns for smooth paths (as done in the main text) extends the variety of the patterns. The concern in appendix D is not whether a pattern is smooth but to generate some interesting patterns by varying mathematical parameters.

For all patterns, except one, in this appendix, $x_{exit} = 3000$, $e_0 = 1$, and $e_N = -1$. Also, not shown, is the circle direction of motion, which is the sign on maximum track angle; any other parameter not shown is 0. Equation (4) does not necessarily hold, which means there can be cusps in the pattern, but equation (3) still holds to give the signed segment length parameter.
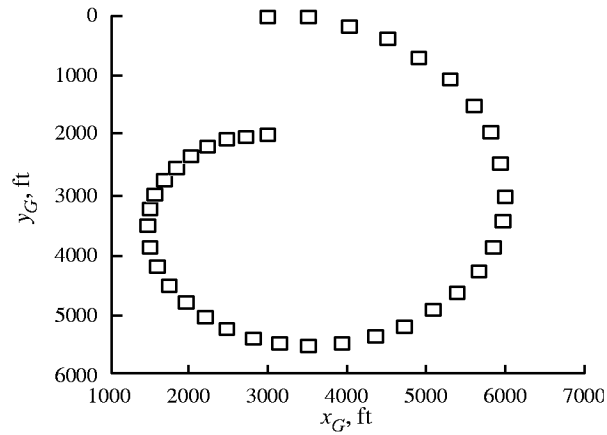
D1. Patterns Using Four 90° Circular Arcs

Each of the three patterns shown in figure D1 uses four 90° circular arcs. The first pattern (fig. D1(a)) is a four-cusp pattern with $N = 4$, $R_0 = R_1 = R_2 = R_3 = 2900$, $(\lambda_1)_{max} = (\lambda_2)_{max} = -90°$, $(\lambda_3)_{max} = (\lambda_4)_{max} = 90°$, and $e_1 = e_2 = e_3 = 1$. The second pattern (fig. D1(b)) shows a two-cusp pattern
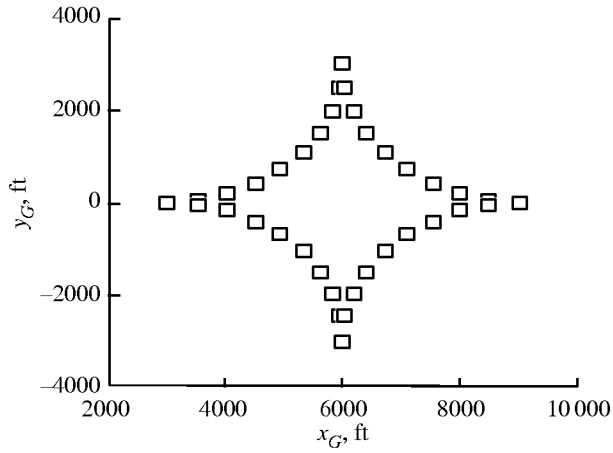
(a) Four cusps.
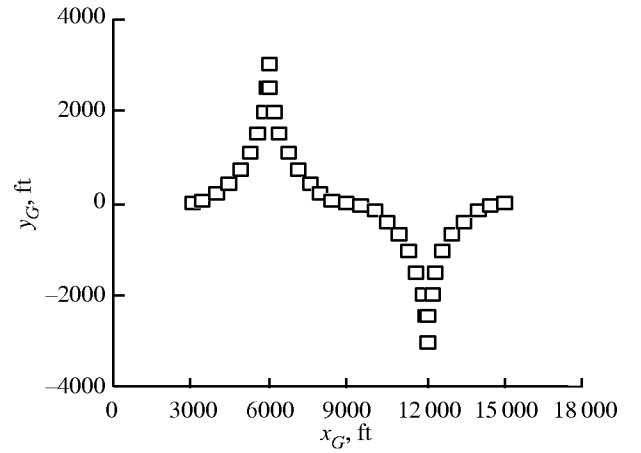
(b) Two cusps.

(c) Spiral.

Figure D1. Patterns using 90° circular arcs.

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resulting from shifting the lower portion of figure D1(a) to the right by letting $l_2 = 6000$. Figure D1(c) is a spiraling pattern, with $N = 4$, $e_1 = e_2 = e_3 = -1$, and $(\lambda_1)_{max} = (\lambda_2)_{max} = (\lambda_3)_{max} = (\lambda_4)_{max} = -90°$.

D2. Square-Wave Pattern

Setting the radius of a circle in a path equal to zero directly connects its tangent segments. For example, with zero radii, the remaining parameters which generate the square wave are $N = 5$, $e_1 = e_2 = e_3 = e_4 = 1$, $(\lambda_1)_{max} = (\lambda_2)_{max} = (\lambda_4)_{max} = -90°$, $(\lambda_3)_{max} = 90°$, $l_1 = l_2 = 1000$, $l_3 = 2000$, and $l_4 = 1000$. This pattern, the square-wave pattern, is shown in figure D2.

D3. Interconnecting Ring Pattern

In the text, the circle classification parameter is either 1 or −1. However, there is no need to impose this restriction in the quest for interesting patterns. For example, to draw two interconnecting rings, let $e_2 = 0$. (This removes all terms that multiply this parameter.) Other parameters for this pattern are $N = 2$, $R_0 = R_1 = 1000$, $(\lambda_1)_{max} = (\lambda_2)_{max} = 360°$, and $l_1 = 1000$. The interconnecting ring pattern is shown in figure D3.
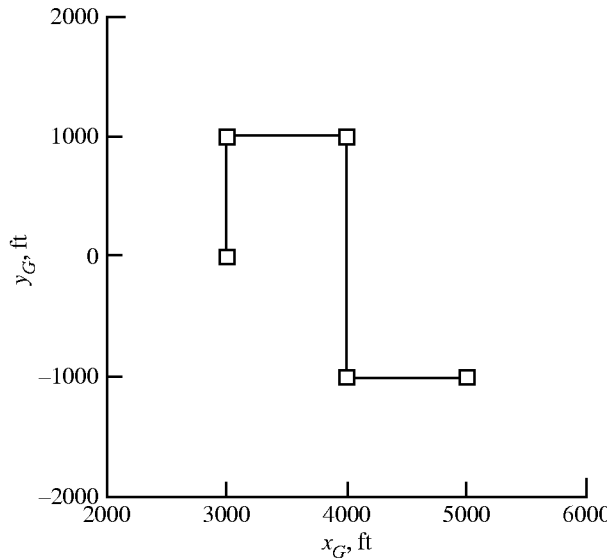

Figure D2. Square-wave pattern.

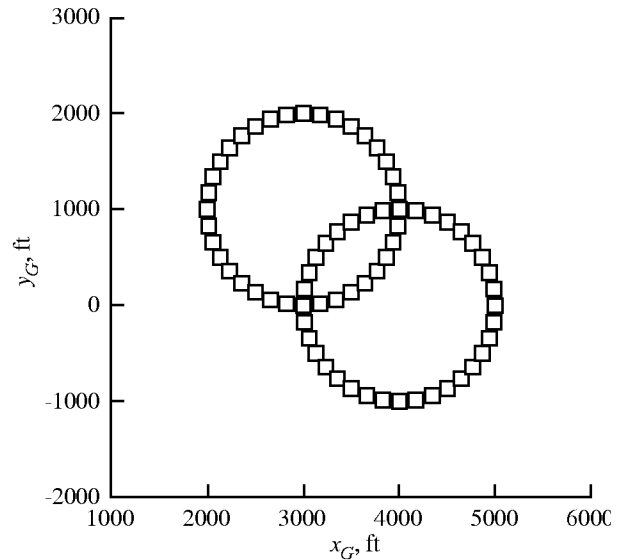Figure D3. Interconnecting ring pattern.

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13. ABSTRACT *(Maximum 200 words)*
The primary purpose of this publication is to develop a mathematical model to describe smooth paths along any combination of circles and tangent lines. Two consecutive circles in a path are either tangent (externally or internally) or they appear on the same (lateral) or opposite (transverse) sides of a connecting tangent line. A path may start or end on either a segment or circle. The approach is to use mathematics common to robotics to design the path as a multilink manipulator. This approach allows a hierarchical view of the problem and keeps the notation manageable. A user simply specifies a few parameters to configure a path. Necessary and sufficient conditions automatically ensure the consistency of the inputs for a smooth path. Two example runway exit paths are given, and an angle to go assists in knowing when to switch from one path element to the next

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